

Extreme paths in oriented 2D Percolation

E. D. Andjel^{a,b} and L. F. Gray^c

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^a Université d'Aix-Marseille, 39 Rue Joliot Curie, 13453 Marseille, France

^b Visiting IMPA, supported by CAPES and CNPq

^c School of Mathematics, University of Minnesota

Abstract

A useful result about leftmost and rightmost paths in two dimensional bond percolation is proved. This result was introduced without proof in [4] in the context of the contact process in continuous time. As discussed here, it also holds for several related models, including the discrete time contact process and two dimensional site percolation. Among the consequences are a natural monotonicity in the probability of percolation between different sites and a somewhat counter-intuitive correlation inequality.

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1 Introduction

The main goal of this paper is to give complete proofs of a result originally presented in [4]. The result was stated for the continuous time contact process in [4], but its proof is missing in the literature. In that paper some interesting consequences are given which we believe justify the writing of the proof here. In the present paper we work in the context of oriented two

dimensional percolation which is equivalent to a discrete time version of the contact process. In the latter part of this paper we discuss how our results apply to other models, and derive some consequences following the ideas of [4].

Two dimensional oriented bond percolation is studied in [3], where some of its most important properties are proved. To introduce the model, let

$$\Lambda = \{(x, y) : x, y \in \mathbb{Z}, y \geq 0, x + y \in 2\mathbb{Z}\}.$$

Then, draw oriented edges from each point (m, n) in Λ to $(m + 1, n + 1)$ and to $(m - 1, n + 1)$. In the percolation literature, the points in Λ and the edges between them are often called sites and bonds, respectively. In this paper we focus on oriented bond percolation, and thus we suppose that the edges are open independently of each other, and that each edge is open with probability $p \in (0, 1)$.

It is an easy matter to adapt the proof here to oriented site percolation, in which the points, rather than the edges, are open with probability p , independently of each other. It turns out that our arguments also continue to work for a more general version of oriented bond percolation in which we allow dependence within each bond pair that emerges from a site. And either by using the so-called “graphical construction” of continuous time interacting particle systems, or by taking limits of discrete time contact processes, our methods can also be applied to various versions of the continuous time contact process in one dimension. For more about such extensions, see our discussion in Section 5.

A path π in Λ is a sequence $(x_0, y_0), \dots, (x_n, y_n)$ of points in Λ such that for all $0 \leq i < n$, $|x_{i+1} - x_i| = 1$ and $y_{i+1} - y_i = 1$. The edges joining (x_i, y_i) to (x_{i+1}, y_{i+1}) for $0 \leq i \leq n - 1$ will be called the edges of π . We say that a path is open if all its edges are open. (In site percolation, a path is open if all its points are open.)

For any $n \in \mathbb{N}_0$ let $L_n = \{(x, n) \in \Lambda\}$. Let $0 \leq m < n$ and let A and B be subsets of L_m and L_n respectively. A path from A to B is any path starting in some point in A and finishing at some point in B . A path π from a point in L_m to a point in L_n will be identified with the function $\pi : [m, n] \cap \mathbb{Z} \rightarrow \mathbb{Z}$ determined by: $(\pi(j), j)$ is a point in the path π for all $m \leq j \leq n$.

Given two paths π_1 and π_2 from L_m to L_n we say that π_1 is to the left of π_2 (or that π_2 is to the right of π_1) and write $\pi_1 \leq \pi_2$ (or $\pi_2 \geq \pi_1$) if $\pi_1(j) \leq \pi_2(j)$ for all $m \leq j \leq n$. This creates a partial order between paths from L_m to L_n . If the inequality is replaced by strict inequality, then we say that π_1 is strictly to the left of π_2 (or π_2 is strictly to the right of π_1).

We find it useful to extend the notions “strictly to the left” and “strictly to the right” to subsets of Λ . Let $P_1 : \Lambda \rightarrow \mathbb{Z}$ be the projection on the first

coordinate: $P_1((x, y)) = x$. And in the usual fashion, extend this function to sets $\Lambda' \subset \Lambda$: $P(\Lambda') = \{P_1((x, y)) : (x, y) \in \Lambda'\}$. For G a subset of Λ , we denote by $\ell(G)$ the set of all points $(j, k) \in \Lambda$ such that $j < \inf P_1(G \cap L_k)$, and we denote by $r(G)$ the set of points $(j, k) \in \Lambda$ such that $j > \sup P_1(G \cap L_k)$. (Here by convention, $\sup \emptyset = -\infty$ and $\inf \emptyset = \infty$.) Thus, $\ell(G)$ ($r(G)$) is the set of all points in Λ that are strictly to the left (right) of G . If G, G' are subsets of Λ , we say that G is strictly to the left of G' or, equivalently, G' is strictly to the right of G , if $G \subset \ell(G')$, or equivalently if $G' \subset r(G)$. And if π is a path, then we say that π is strictly to the left of G (strictly to the right of G) if the set of points in π is strictly to the left of G (strictly to the right of G). The notation $\ell(\cdot)$ and $r(\cdot)$ introduced here also applies to paths, thinking of them as sets. For example, a path π is strictly to the left of a set G if and only if $G \subset r(\pi)$. Please note that this terminology and notation are consistent with our earlier definition of one path being strictly to the left of another path, but that they now also apply to paths that do not necessarily start on the same level L_m or end on the same level L_n .

Let $\Lambda' \subset \Lambda$. Note that, if $A \subset L_m$ and $B \subset L_n$ are finite and there is at least one path from A to B contained in Λ' , then there is a unique path from A to B contained in Λ' which is to the left of all paths from A to B contained in Λ' . This is called the leftmost path from A to B . And, if there is an open path from A to B contained in Λ' , then there is a unique open path from A to B contained in Λ' which is to the left of all open paths from A to B contained in Λ' . This path will be called the leftmost open path from A to B in Λ' . Similarly, we define the rightmost path and rightmost open path from A to B in Λ' .

Given a subset Λ' of Λ , $0 \leq m < n \in \mathbb{N}$ and two finite subsets A and B of L_m and L_n respectively, $\Gamma_{\Lambda'}(A, B)$ will denote the set of paths from A to B contained in Λ' . If this set is non-empty, then $\mu_{\Lambda'}(A, B)$ ($\nu_{\Lambda'}(A, B)$) will denote the conditional distribution of the leftmost (rightmost) open path from A to B contained in Λ' given that there is at least one open path from A to B in Λ' . If $m \leq j \leq n$ and $C \subset L_j \cap \Lambda'$, $\Gamma_{\Lambda'}(A, C, B)$ will denote the set of paths from A to B going through a point in C and contained in Λ' , and if this set is non-empty, then $\mu_{\Lambda'}(A, C, B)$ ($\nu_{\Lambda'}(A, C, B)$) will denote the conditional distribution of the leftmost (rightmost) open path from A to B contained in Λ' and going through a point in C given that there is at least one such open path. In all these notations the subscript Λ' will be omitted if Λ' is the whole set Λ , and when either A, B or C is a singleton, say $\{(x, y)\}$, we will often write (x, y) rather than $\{(x, y)\}$. Finally, if $\gamma_1 \in \Gamma(A, (x, y))$ and $\gamma_2 \in \Gamma((x, y), B)$ then $\gamma_1 \gamma_2 \in \Gamma(A, (x, y), B)$ will be the concatenation of γ_1 and γ_2 . When the coordinates are not important, we will often denote a point $(x, y) \in \Lambda$ as a single boldface letter, such as $\mathbf{z} = (x, y)$, so that, for

example, $\Gamma(A, (x, y), B)$ might be written as $\Gamma(A, \mathbf{z}, B)$.

Let $0 \leq m < n$, let A and B be subsets of L_m and L_n respectively such that $\Gamma(A, B)$ is nonempty, and let μ and ν be probability measures on $\Gamma(A, B)$. We say that μ is stochastically to the left of ν and write $\mu \leq \nu$ if for any increasing function Φ on $\Gamma(A, B)$ we have

$$\int_{\Gamma(A, B)} \Phi(\gamma) d\mu(\gamma) \leq \int_{\Gamma(A, B)} \Phi(\gamma) d\nu(\gamma).$$

We can now state our version of the main result in [4]:

Theorem 1.1 *Let $0 \leq m < n$, let A and B be finite subsets of L_m and L_n respectively and let G be a subset of Λ . If $\Gamma_{\ell(G)}(A, B)$ is nonempty, we have $\mu_{\ell(G)}(A, B) \leq \mu(A, B)$ and $\nu_{\ell(G)}(A, B) \leq \nu(A, B)$. And if $\Gamma_{r(G)}(A, B)$ is nonempty, we have $\mu_{r(G)}(A, B) \geq \mu(A, B)$ and $\nu_{r(G)}(A, B) \geq \nu(A, B)$.*

This result has the following immediate corollary:

Corollary 1.1 *Let $m < n$, let A and B be finite subsets of L_m and L_n respectively such that $\Gamma(A, B)$ is nonempty and suppose \mathbf{a} is a point in L_m that is strictly to the right of A . Then, $\mu(A \cup \{\mathbf{a}\}, B) \geq \mu(A, B)$ and $\nu(A \cup \{\mathbf{a}\}, B) \geq \nu(A, B)$. Moreover, if \mathbf{b} is a point in L_n that is strictly to the right of B , then $\mu(A, B \cup \{\mathbf{b}\}) \geq \mu(A, B)$ and $\nu(A, B \cup \{\mathbf{b}\}) \geq \nu(A, B)$. If, instead \mathbf{a} is strictly to the left of A (\mathbf{b} is strictly to the left of B) then the first two (last two) inequalities are reversed.*

The different parts of the corollary follow from the theorem by making appropriate choices of the sets A, B, G . For example, for the first part of the corollary, replace A in the theorem by $A \cup \{\mathbf{a}\}$ and let $G = \{\mathbf{a}\}$.

In Section 2 we prove two elementary lemmas. Then, in Section 3 we prove a key proposition and with it in hand we prove the theorem, using a somewhat involved inductive argument that was hinted at in [4]. In Section 4, we give an alternate proof of our main theorem, based on a Markov chain that was introduced in [2]. We discovered this approach after we had fully developed our inductive argument. The Markov chain argument is shorter, but we have not been able to generalize it to other models quite as well as our inductive argument. Finally, in the last two sections, we discuss extensions to other models and derive some consequences of the theorem and its corollary following the ideas of [4].

2 Basic lemmas

We start this section with a very simple lemma:

Lemma 2.1 *Let $m < n$ and let G be a non-empty subset of Λ such that the projection $P_1(G)$ is bounded below. Then, there exists a path τ_G from L_m to L_n such that any other path γ from L_m to L_n is strictly to the left of G if and only if it is strictly to the left of τ_G . If instead $P_1(G)$ is bounded above, the same statement holds if we substitute right for left.*

Proof of Lemma 2.1. Suppose $P_1(G)$ is bounded below. For each $(x, y) \in G$, let $B_{x,y} = \{(u, v) \in \Lambda : u \geq x + |v - y|\}$ and let $B(G) = \cup_{(x,y) \in G} B_{x,y}$. Then for $m \leq j \leq n$ define $x_j = \inf P_1(B(G) \cap L_j)$. It is now easy to verify that the sequence $(x_j, j) : m \leq j \leq n$ defines a path τ_G from L_m to L_n satisfying the conclusion of the lemma. A similar proof works when $P_1(G)$ is bounded above. \square

Before stating our next lemma, we introduce some further notation: Fix integers $0 \leq m < j < n$ and sets $A \subset L_m, B \subset L_n, C \subset L_j$. Given a probability measure ρ on $\Gamma(A, C, B)$ we call ρ_1 and ρ_2 its marginals on $\Gamma(A, C)$ and $\Gamma(C, B)$ respectively. They are given by:

$$\rho_1(\gamma_1) = \sum_{\gamma_2 \in \Gamma(C, B)} \rho(\gamma_1 \gamma_2) \text{ and}$$

$$\rho_2(\gamma_2) = \sum_{\gamma_1 \in \Gamma(A, C)} \rho(\gamma_1 \gamma_2).$$

Given $\gamma_1 \in \Gamma(A, C)$ such that $\rho_1(\gamma_1) > 0$, we define the conditional measure $\rho(\bullet | \gamma_1)$ on $\Gamma(C, B)$ by:

$$\rho(\gamma_2 | \gamma_1) = \frac{\rho(\gamma_1 \gamma_2)}{\rho_1(\gamma_1)}.$$

We note that we have perhaps abused the conditional probability notation here slightly, since it may seem more technically accurate to let $\rho(\bullet | \gamma_1)$ denote a probability measure on paths from A to B (with the portion from A to C equaling γ_1), rather than the way we have defined it, which is as a probability measure on paths from C to B , but we hope that this abuse will not cause any confusion for the reader.

The following lemma is needed to prove the key proposition stated in the next section.

Lemma 2.2 *Let m, j and n be integers such that $0 \leq m < j < n$. Then, let $A \subset L_m$ and $B \subset L_n$ be finite and let $\mathbf{a} \in L_j$ be such that $\Gamma(A, \mathbf{a})$ and $\Gamma(\mathbf{a}, B)$ are nonempty. Finally, let ρ_1 and ρ_2 be probability measures on $\Gamma(A, \mathbf{a})$ and*

on $\Gamma(\mathbf{a}, B)$ respectively and let ρ be the probability measure on $\Gamma(A, \mathbf{a}, B)$ defined by $\rho(\gamma_1\gamma_2) = \rho_1(\gamma_1)\rho_2(\gamma_2)$.

If σ is a probability measure on $\Gamma(A, \mathbf{a}, B)$ such that

i) $\rho_1 \leq \sigma_1$.

ii) $\rho_2 \leq \sigma(\bullet|\gamma_1)$ for any $\gamma_1 \in \Gamma(A, \mathbf{a})$ such that $\sigma_1(\gamma_1) > 0$,
then $\rho \leq \sigma$.

Proof of Lemma 2.2. Let ϕ be increasing on $\Gamma(A, \mathbf{a}, B)$. Then write

$$\begin{aligned} \int_{\Gamma(A, \mathbf{a}, B)} \phi(\gamma) d\sigma(\gamma) &= \\ \sum_{\gamma_1 \in \Gamma(A, \mathbf{a})} \left(\sum_{\gamma_2 \in \Gamma(\mathbf{a}, B)} \phi(\gamma_1\gamma_2) \sigma(\gamma_2|\gamma_1) \right) \sigma_1(\gamma_1) &\geq \\ \sum_{\gamma_1 \in \Gamma(A, \mathbf{a})} \left(\sum_{\gamma_2 \in \Gamma(\mathbf{a}, B)} \phi(\gamma_1\gamma_2) \rho_2(\gamma_2) \right) \sigma_1(\gamma_1) &\geq \\ \sum_{\gamma_1 \in \Gamma(A, \mathbf{a})} \left(\sum_{\gamma_2 \in \Gamma(\mathbf{a}, B)} \phi(\gamma_1\gamma_2) \rho_2(\gamma_2) \right) \rho_1(\gamma_1) &= \\ \int_{\Gamma(A, \mathbf{a}, B)} \phi(\pi) d\rho(\pi), \end{aligned}$$

where the first inequality follows from the fact that for all $\gamma_1 \in \Gamma(A, \mathbf{a})$ $\sigma(\bullet|\gamma_1) \geq \rho_2$ and $\gamma_2 \rightarrow \phi(\gamma_1\gamma_2)$ is an increasing function and the second inequality follows from the fact that $\sigma_1 \geq \rho_1$ and

$$\gamma_1 \rightarrow \sum_{\gamma_2 \in \Gamma(\mathbf{a}, B)} \phi(\gamma_1\gamma_2) \rho_2(\gamma_2)$$

is an increasing function. □

3 Proof of Theorem 1.1

We have defined $\mu(A, B)$ as the conditional distribution of the leftmost open path from A to B given the event

$$H = \{\text{there exists an open path from } A \text{ to } B\}.$$

In this section we adopt the following notation: if F is another event such that $P(F \cap H) > 0$, then $\mu^F(A, B)$ is the distribution of the leftmost open

path from A to B given the event $F \cap H$. The same notation will apply to distributions such as $\mu(A, C, B)$, $\mu_{\Lambda_0}(A, B)$ etc. and to distributions of rightmost open paths such as $\nu(A, B)$, $\nu(A, C, B)$, $\nu_{\Lambda_0}(A, B)$ etc.

We now extend the notion of paths from L_m to L_n by adding two extra paths: the sequence $(-\infty, m), \dots, (-\infty, n)$ and the sequence $(\infty, m), \dots, (\infty, n)$. We often suppress m and n and simply denote these paths by $-\infty$ and ∞ respectively. The path $-\infty$ (∞) will be considered as being strictly to the left (right) of any other path from L_m to L_n , and also of any subset of Λ . Finally, given $0 \leq m < n$ and two paths τ_1 and τ_2 from L_m to L_n such that τ_1 is strictly to the left of τ_2 we let $b(\tau_1, \tau_2)$ be the set of points that are strictly to the right of τ_1 and strictly to the left of τ_2 .

We now state a proposition which is a slightly weaker version of Theorem 1.1, namely:

Proposition 3.1 *Let $0 \leq m < n$, let τ_1, τ_2 and τ_3 be paths from L_m to L_n such that τ_1 is to the left of τ_3 and τ_3 is to the left of τ_2 , and let A and B be non-empty finite subsets of L_m and L_n respectively. If A is strictly to the right of τ_3 and there exists a path from A to B in $b(\tau_3, \tau_2)$, then $\mu_{b(\tau_1, \tau_2)}(A, B) \leq \mu_{b(\tau_3, \tau_2)}(A, B)$ and $\nu_{b(\tau_1, \tau_2)}(A, B) \leq \nu_{b(\tau_3, \tau_2)}(A, B)$. Similarly, if A is strictly to the left of τ_3 and there exists a path from A to B in $b(\tau_1, \tau_3)$ we have $\mu_{b(\tau_1, \tau_3)}(A, B) \leq \mu_{b(\tau_1, \tau_2)}(A, B)$ and $\nu_{b(\tau_1, \tau_3)}(A, B) \leq \nu_{b(\tau_1, \tau_2)}(A, B)$.*

The proof of this proposition requires a rather involved inductive argument. To facilitate its reading, we start deriving some consequences of the inductive hypothesis of that proof. That is the purpose of our next lemma and its corollaries.

Lemma 3.1 *Suppose that Proposition 3.1 holds for $m = 0$ and some $n > 0$. Let τ_1 and τ_2 be paths from L_0 to L_n such that τ_1 is strictly to the left of τ_2 , let A and B be finite subsets of L_0 and L_n respectively, and let \mathbf{a} be a point in L_0 and \mathbf{b} a point in L_n . Assuming there exists a path in $b(\tau_1, \tau_2)$ from A to B , the following statements hold:*

i) If \mathbf{a} is strictly to the right of A then,

$$\mu_{b(\tau_1, \tau_2)}(A, B) \leq \mu_{b(\tau_1, \tau_2)}(A \cup \{\mathbf{a}\}, B) \text{ and}$$

$$\nu_{b(\tau_1, \tau_2)}(A, B) \leq \nu_{b(\tau_1, \tau_2)}(A \cup \{\mathbf{a}\}, B).$$

ii) If \mathbf{b} is strictly to the right of B then,

$$\mu_{b(\tau_1, \tau_2)}(A, B) \leq \mu_{b(\tau_1, \tau_2)}(A, B \cup \{\mathbf{b}\}) \text{ and}$$

$$\nu_{b(\tau_1, \tau_2)}(A, B) \leq \nu_{b(\tau_1, \tau_2)}(A, B \cup \{\mathbf{b}\}).$$

If instead, \mathbf{a} is strictly to the left of A the first two inequalities are reversed while if \mathbf{b} is strictly to the left of B the last two are reversed.

Proof of Lemma 3.1. We only prove the first inequality since the proofs of the others are similar. We may assume that there exists a path from \mathbf{a} to B between τ_1 and τ_2 , since otherwise the result is trivial. Let

$$F_A = \{ \text{there is no open path in } b(\tau_1, \tau_2) \text{ from } A \text{ to } B \},$$

let

$$F_{\mathbf{a}} = \{ \text{there is no open path in } b(\tau_1, \tau_2) \text{ from } \mathbf{a} \text{ to } B \},$$

and let Φ be the random set of edges belonging to open paths starting from A and contained in $b(\tau_1, \tau_2)$. Then the event F_A is the union of disjoint events of the form $\{\Phi = \varphi\}$ where φ ranges over a collection Υ of deterministic sets of edges, and $\mu_{b(\tau_1, \tau_2)}^{F_A}(\mathbf{a}, B)$ is a convex combination of measures of the form $\mu_{b(\tau_1, \tau_2)}^{\{\Phi = \varphi\}}(\mathbf{a}, B)$ where φ is such that there exists a path from \mathbf{a} to B in $b(\tau_1, \tau_2)$ which is strictly to the right of φ . Now, we let $V(\varphi)$ be the set of points which are vertices of edges in φ and observe that on the event $\{\Phi = \varphi\}$ any open path from \mathbf{a} to B must be strictly to the right of $V(\varphi)$. By Lemma 2.1 there exists a path $\tau(\varphi)$ such that the paths which are strictly to the right of $V(\varphi)$ are exactly those which are strictly to the right of $\tau(\varphi)$. It then follows that

$$\mu_{b(\tau_1, \tau_2)}^{\{\Phi = \varphi\}}(\mathbf{a}, B) = \mu_{b(\tau_1(\varphi), \tau_2)}(\mathbf{a}, B), \quad (1)$$

where $\tau_1(\varphi)$ is the leftmost path such that $\tau_1 \leq \tau_1(\varphi)$ and $\tau(\varphi) \leq \tau_1(\varphi)$. We have used the fact that \mathbf{a} is necessarily strictly between $\tau_1(\varphi)$ and τ_2 . We also needed the following: given $\{\Phi = \varphi\}$, the conditional distribution of the openness of the bonds emanating from sites that are strictly to the right of $\tau_1(\varphi)$ is the same as the unconditional distribution. This fact follows easily from the independence that is built into the model.

Since we are assuming that the conclusion of Proposition 3.1 holds we obtain:

$$\mu_{b(\tau_1, \tau_2)}(\mathbf{a}, B) \leq \mu_{b(\tau_1, \tau_2)}^{\{\Phi = \varphi\}}(\mathbf{a}, B).$$

Therefore,

$$\mu_{b(\tau_1, \tau_2)}(\mathbf{a}, B) \leq \mu_{b(\tau_1, \tau_2)}^{F_A}(\mathbf{a}, B). \quad (2)$$

Since $\mu_{b(\tau_1, \tau_2)}(\mathbf{a}, B)$ is a convex combination of $\mu_{b(\tau_1, \tau_2)}^{F_A}(\mathbf{a}, B)$ and of $\mu_{b(\tau_1, \tau_2)}^{F_A^c}(\mathbf{a}, B)$, (2) implies:

$$\mu_{b(\tau_1, \tau_2)}^{F_A^c}(\mathbf{a}, B) \leq \mu_{b(\tau_1, \tau_2)}(\mathbf{a}, B). \quad (3)$$

Similarly one shows that

$$\mu_{b(\tau_1, \tau_2)}^{F_{\mathbf{a}}}(\mathbf{a}, B) \leq \mu_{b(\tau_1, \tau_2)}(\mathbf{a}, B). \quad (4)$$

and that

$$\mu_{b(\tau_1, \tau_2)}(A, B) \leq \mu_{b(\tau_1, \tau_2)}^{F_{\mathbf{a}}^c}(A, B). \quad (5)$$

But on the event $F_A^c \cap F_{\mathbf{a}}^c$ the leftmost path from A to B is to the left of the leftmost path from \mathbf{a} to B , hence

$$\mu_{b(\tau_1, \tau_2)}^{F_{\mathbf{a}}^c}(A, B) = \mu_{b(\tau_1, \tau_2)}^{F_A^c \cap F_{\mathbf{a}}^c}(A, B) \leq \mu_{b(\tau_1, \tau_2)}^{F_A^c \cap F_{\mathbf{a}}^c}(\mathbf{a}, B) = \mu_{b(\tau_1, \tau_2)}^{F_A^c}(\mathbf{a}, B). \quad (6)$$

It now follows from (3), (5) and (6) that:

$$\mu_{b(\tau_1, \tau_2)}(A, B) \leq \mu_{b(\tau_1, \tau_2)}(\mathbf{a}, B). \quad (7)$$

Since on the event F_A^c the leftmost path from $A \cup \{\mathbf{a}\}$ to B is the same as the leftmost path from A to B we have:

$$\mu_{b(\tau_1, \tau_2)}^{F_A^c}(A \cup \{\mathbf{a}\}, B) = \mu_{b(\tau_1, \tau_2)}(A, B). \quad (8)$$

And, since on the event F_A the leftmost path from $A \cup \{\mathbf{a}\}$ to B is the same as the leftmost path from \mathbf{a} to B we also have:

$$\mu_{b(\tau_1, \tau_2)}^{F_A}(A \cup \{\mathbf{a}\}, B) = \mu_{b(\tau_1, \tau_2)}^{F_A}(\mathbf{a}, B). \quad (9)$$

Therefore $\mu_{b(\tau_1, \tau_2)}(A \cup \{\mathbf{a}\}, B)$ is a convex combination of $\mu_{b(\tau_1, \tau_2)}(A, B)$ and $\mu_{b(\tau_1, \tau_2)}^{F_A}(\mathbf{a}, B)$. But, it follows from (2) and (7) that

$$\mu_{b(\tau_1, \tau_2)}(A, B) \leq \mu_{b(\tau_1, \tau_2)}^{F_A}(\mathbf{a}, B).$$

Therefore

$$\mu_{b(\tau_1, \tau_2)}(A, B) \leq \mu_{b(\tau_1, \tau_2)}(A \cup \{\mathbf{a}\}, B).$$

□

Corollary 3.1 *Assume the hypothesis of Lemma 3.1 and let A' be a finite subset of L_0 . If A' is strictly to the right of A , then*

$$\mu_{b(\tau_1, \tau_2)}(A, B) \leq \mu_{b(\tau_1, \tau_2)}(A \cup A', B).$$

If in addition there exists a path in $b(\tau_1, \tau_2)$ from A' to B then,

$$\mu_{b(\tau_1, \tau_2)}(A, B) \leq \mu_{b(\tau_1, \tau_2)}(A', B).$$

If A' is strictly to the left of A , then

$$\mu_{b(\tau_1, \tau_2)}(A \cup A', B) \leq \mu_{b(\tau_1, \tau_2)}(A, B).$$

If in addition there exists a path in $b(\tau_1, \tau_2)$ from A' to B then

$$\mu_{b(\tau_1, \tau_2)}(A', B) \leq \mu_{b(\tau_1, \tau_2)}(A, B).$$

Proof of Corollary 3.1. Assume A' is strictly to the right of A . Then, let $\mathbf{a}'_1 < \mathbf{a}'_2, \dots, \mathbf{a}'_s$ be the points of A' and let $A'_j = \{\mathbf{a}'_1, \dots, \mathbf{a}'_j\}$. It then follows from Lemma 3.1 that

$$\mu_{b(\tau_1, \tau_2)}(A, B) \leq \mu_{b(\tau_1, \tau_2)}(A \cup A'_1, B) \leq \dots \leq \mu_{b(\tau_1, \tau_2)}(A \cup A'_s, B),$$

and the first inequality is proved. The third inequality is proved in the same way. Then using the third inequality interchanging the roles of A and A' we get

$$\mu_{b(\tau_1, \tau_2)}(A \cup A', B) \leq \mu_{b(\tau_1, \tau_2)}(A', B),$$

and together with the first inequality, this implies the second inequality. The fourth inequality is proved following the same method. \square

In the same way we also get the following two corollaries:

Corollary 3.2 *Assume the hypothesis of Lemma 3.1 and let B' be a finite subset of L_n . If B' is strictly to the right of B , then*

$$\mu_{b(\tau_1, \tau_2)}(A, B) \leq \mu_{b(\tau_1, \tau_2)}(A, B \cup B').$$

If in addition there exists a path in $b(\tau_1, \tau_2)$ from A' to B then,

$$\mu_{b(\tau_1, \tau_2)}(A, B) \leq \mu_{b(\tau_1, \tau_2)}(A, B').$$

If B' is strictly to the left of B , then

$$\mu_{b(\tau_1, \tau_2)}(A, B \cup B') \leq \mu_{b(\tau_1, \tau_2)}(A, B).$$

If in addition there exists a path in $b(\tau_1, \tau_2)$ from A to B' then

$$\mu_{b(\tau_1, \tau_2)}(A, B') \leq \mu_{b(\tau_1, \tau_2)}(A, B).$$

Corollary 3.3 *Corollaries 3.1 and 3.2 also hold if in their statements we substitute distributions of rightmost open paths for distributions of leftmost open paths (i.e. the inequalities also hold if we write ν instead of μ).*

Proof of Proposition 3.1.. Without loss of generality, we assume that $m = 0$ and proceed by induction on n , the inductive statement being that the four inequalities hold under their respective hypotheses. For $n = 1$ the result is trivial. For the inductive step, we wish to prove that the four inequalities hold under their respective hypotheses for a specific value of $n \geq 2$, given the

inductive hypothesis, which is that the four inequalities all hold under their respective hypotheses for all smaller values of $n \geq 1$.

We will prove that

$$\mu_{b(\tau_1, \tau_3)}(A, B) \leq \mu_{b(\tau_1, \tau_2)}(A, B) \quad (10)$$

where A and B are subsets of L_0 and L_n respectively and τ_1, τ_2, τ_3 are paths from L_0 to L_n satisfying the appropriate hypotheses, including the hypothesis that A is strictly to the left of τ_3 . We omit the proofs of the other inequalities because they follow the same ideas. Let $\mathbf{a}_0, \dots, \mathbf{a}_n$ be the successive vertices of τ_3 and for $i = 0, \dots, n$, let

$$\Lambda_i = \ell(\{\mathbf{a}_0, \dots, \mathbf{a}_i\}) \cap b(\tau_1, \tau_2).$$

This is the set of vertices that are strictly between τ_1 and τ_2 and also strictly to the left of the first $i + 1$ vertices of τ_3 .

Of course, $\Lambda_n = b(\tau_1, \tau_3)$, so $\mu_{b(\tau_1, \tau_3)}(A, B) = \mu_{\Lambda_n}(A, B)$. Since we have assumed that A is strictly to the left of τ_3 , we also have $\mu_{\Lambda_0}(A, B) = \mu_{b(\tau_1, \tau_2)}(A, B)$. Therefore, the result will follow from:

$$\mu_{\Lambda_{i+1}}(A, B) \leq \mu_{\Lambda_i}(A, B) \quad i = 0, \dots, n-1. \quad (11)$$

We fix an $i \in \{0, \dots, n-1\}$ and note that $\mu_{\Lambda_{i+1}}(A, B) = \mu_{\Lambda_i}(A, B)$ if the paths from A to B in Λ_{i+1} and in Λ_i are the same. These sets of paths are the same if there is no path from A to B that goes through \mathbf{a}_{i+1} , or if $\mathbf{a}_{i+1} = \mathbf{a}_i + (1, 1)$. Thus, in proving (11), we may assume that there is at least one path from A to B going through \mathbf{a}_{i+1} and that $\mathbf{a}_{i+1} = \mathbf{a}_i + (-1, 1)$.

We start proving (11) for $i \in \{0, \dots, n-2\}$. Later, we will show it also holds for $i = n-1$. Let

$$F = \{\text{there is no open path from } A \text{ to } B \text{ in } \Lambda_{i+1}\}. \quad (12)$$

We now show that

$$\mu_{\Lambda_i}(A, \mathbf{a}_{i+1}, B) \leq \mu_{\Lambda_i}^F(A, \mathbf{a}_{i+1}, B). \quad (13)$$

In words, this says that the leftmost open path from A to B contained in Λ_i and passing through \mathbf{a}_{i+1} is stochastically to the left of the leftmost open path from A to B contained in Λ_i and passing through \mathbf{a}_{i+1} given that there is no open path from A to B in Λ_{i+1} . To prove (13) we will show that $\mu_{\Lambda_i}(A, \mathbf{a}_{i+1}, B)$ and $\mu_{\Lambda_i}^F(A, \mathbf{a}_{i+1}, B)$ satisfy the hypothesis of Lemma 2.2. Clearly

$$\mu_{\Lambda_i}(A, \mathbf{a}_{i+1}, B) = \mu_{\Lambda_i}(A, \mathbf{a}_{i+1}) \times \mu_{\Lambda_i}(\mathbf{a}_{i+1}, B)$$

Hence, to show (13) it suffices to show that the measures of this inequality satisfy the hypotheses i) and ii) of Lemma 2.2 .

Let Φ be the random set of edges that belong to open paths in Λ_i starting from points in A which do not go through \mathbf{a}_{i+1} (but may have \mathbf{a}_{i+1} as their endpoint). Then the event F can be expressed as the union of disjoint events of the form $\{\Phi = \varphi\}$ where φ ranges over some deterministic subsets of edges. We call Υ the collection of those subsets: $F = \cup_{\varphi \in \Upsilon} \{\Phi = \varphi\}$. Fix a path γ_1 from A to \mathbf{a}_{i+1} in Λ_i and let Υ_{γ_1} be the collection of sets in Υ which contain all the edges of γ_1 and contain no other path from A to \mathbf{a}_{i+1} in Λ_i to the left of γ_1 . Note that for any $\varphi \in \Upsilon_{\gamma_1}$, on the event $\{\Phi = \varphi\}$ any open path from \mathbf{a}_{i+1} to B must be strictly to the right of φ . Now, $\mu_{\Lambda_i}^F(A, \mathbf{a}_{i+1}, B)(\bullet|\gamma_1)$ is a convex combination of measures of the form $\mu_{\Lambda_i}^{\{\Phi=\varphi\}}(A, \mathbf{a}_{i+1}, B)_2$ where φ ranges over Υ_{γ_1} . But for each φ , the corresponding measure is equal to $\mu_{\Lambda_i \cap r(\varphi)}(\mathbf{a}_{i+1}, B)$ i.e. the distribution of the leftmost open path from \mathbf{a}_{i+1} to B in Λ_i which is strictly to the right of φ . In making this assertion, we are using the fact that given $\{\Phi = \varphi\}$, the conditional distribution of the openness of the bonds that emerge from sites that are strictly to the right of φ is the same as the unconditional distribution. This consequence of the independence in the model is similar to the one we used in the proof of Lemme 3.1 when we proved (1). We note for future reference (see Section 5) that both here and in the proof of Lemma 3.1, we do not need independence for two bonds that emerge from the same site; we only need it for bonds that emerge from different sites.

By the inductive hypothesis, and Lemma 2.1 we have

$$\mu_{\Lambda_i \cap r(\varphi)}(\mathbf{a}_{i+1}, B) \geq \mu_{\Lambda_i}(\mathbf{a}_{i+1}, B),$$

for any $\varphi \in \Upsilon_{\gamma_1}$. Hence

$$\begin{aligned} \mu_{\Lambda_i}^F(A, \mathbf{a}_{i+1}, B)(\bullet|\gamma_1) &\geq \mu_{\Lambda_i}(\mathbf{a}_{i+1}, B) = \\ &\mu_{\Lambda_i}(A, \mathbf{a}_{i+1}, B)_2. \end{aligned}$$

Since this holds for all γ_1 , it implies that

$$\mu_{\Lambda_i}^F(A, \mathbf{a}_{i+1}, B)_2 \geq \mu_{\Lambda_i}(A, \mathbf{a}_{i+1}, B)_2.$$

Since $i \leq n-2$ we can use the inductive hypothesis in the same way to show that

$$\mu_{\Lambda_i}^F(A, \mathbf{a}_{i+1}, B)_1 \geq \mu_{\Lambda_i}(A, \mathbf{a}_{i+1}, B)_1. \quad (14)$$

The one important difference in this argument is that we must partition the event F according to open paths that end at the set B instead of starting at A . Now (13) follows from Lemma 2.2. Since

$$\mu_{\Lambda_i}^F(A, B) = \mu_{\Lambda_i}^F(A, \mathbf{a}_{i+1}, B),$$

from (13) we get:

$$\mu_{\Lambda_i}^F(A, B) \geq \mu_{\Lambda_i}(A, \mathbf{a}_{i+1}, B).$$

And since

$$\mu_{\Lambda_i}^{F^c}(A, B) = \mu_{\Lambda_{i+1}}(A, B),$$

(11) will follow if we show that:

$$\mu_{\Lambda_i}(A, \mathbf{a}_{i+1}, B) \geq \mu_{\Lambda_{i+1}}(A, B). \quad (15)$$

To prove this, we let

$$C_j = \{\mathbf{a}_{i+1} - (2j, 0), \dots, \mathbf{a}_{i+1} - (2, 0)\},$$

where j ranges from \underline{j} to \bar{j} , where \underline{j} is the lowest value of j for which $\Gamma_{\Lambda_{i+1}}(A, C_j, B)$ is nonempty and \bar{j} is the largest value of j for which C_j is strictly to the right of τ_1 . We will now prove by induction on j that

$$\mu_{\Lambda_i}(A, \mathbf{a}_{i+1}, B) \geq \mu_{\Lambda_{i+1}}(A, C_j, B), \quad (16)$$

for all $\underline{j} \leq j \leq \bar{j}$. Since $\mu_{\Lambda_{i+1}}(A, B) = \mu_{\Lambda_{i+1}}(A, C_{\bar{j}}, B)$, (15) will follow. For $j = \underline{j}$, first note that

$$\mu_{\Lambda_{i+1}}(A, C_{\underline{j}}, B) = \mu_{\Lambda_{i+1}}(A, \mathbf{a}_{i+1} - (2\underline{j}, 0)) \times \mu_{\Lambda_{i+1}}(\mathbf{a}_{i+1} - (2\underline{j}, 0), B)$$

and

$$\mu_{\Lambda_i}(A, \mathbf{a}_{i+1}, B) = \mu_{\Lambda_i}(A, \mathbf{a}_{i+1}) \times \mu_{\Lambda_i}(\mathbf{a}_{i+1}, B).$$

Since

$$\mu_{\Lambda_{i+1}}(A, \mathbf{a}_{i+1} - (2\underline{j}, 0)) = \mu_{\Lambda_i}(A, \mathbf{a}_{i+1} - (2\underline{j}, 0)),$$

from Corollary 3.2 and the inductive hypothesis on n we get:

$$\mu_{\Lambda_{i+1}}(A, \mathbf{a}_{i+1} - (2\underline{j}, 0)) \leq \mu_{\Lambda_i}(A, \mathbf{a}_{i+1}).$$

Similarly, since

$$\mu_{\Lambda_{i+1}}(\mathbf{a}_{i+1} - (2\underline{j}, 0), B) = \mu_{\Lambda_i}(\mathbf{a}_{i+1} - (2\underline{j}, 0), B),$$

from Corollary 3.1 and the same inductive hypothesis we get:

$$\mu_{\Lambda_{i+1}}(\mathbf{a}_{i+1} - (2\underline{j}, 0), B) \leq \mu_{\Lambda_i}(\mathbf{a}_{i+1}, B).$$

Therefore for $j = \underline{j}$, (16) follows from Lemma 2.2. To prove the inductive step, we assume that (16) holds for some $\underline{j} < j < \bar{j}$. We also assume that

$\Gamma_{\Lambda_{i+1}}(A, \mathbf{a}_{i+1} - (2(j+1), 0), B)$ is nonempty since otherwise the inductive step is trivial. Then, we define the event:

$$F_j = \{\text{there is no open path from } A \text{ to } B \text{ in } \Lambda_{i+1} \text{ going through } C_j\}.$$

Since

$$\mu_{\Lambda_{i+1}}^{F_j^c}(A, C_{j+1}, B) \leq \mu_{\Lambda_{i+1}}(A, C_j, B)$$

and $\mu_{\Lambda_{i+1}}(A, C_{j+1}, B)$ is a convex combination of $\mu_{\Lambda_{i+1}}^{F_j^c}(A, C_{j+1}, B)$ and $\mu_{\Lambda_{i+1}}^{F_j}(A, C_{j+1}, B)$,

$$\mu_{\Lambda_{i+1}}(A, C_{j+1}, B) \leq \mu_{\Lambda_i}(A, \mathbf{a}_{i+1}, B)$$

will follow from the inductive hypothesis (in j) and

$$\mu_{\Lambda_{i+1}}^{F_j}(A, C_{j+1}, B) \leq \mu_{\Lambda_i}(A, \mathbf{a}_{i+1}, B). \quad (17)$$

To prove (17) we start noting that

$$\mu_{\Lambda_{i+1}}^{F_j}(A, C_{j+1}, B) = \mu_{\Lambda_{i+1}}^{F_j}(A, \mathbf{a}_{i+1} - (2(j+1), 0), B).$$

We also claim that

$$\mu_{\Lambda_{i+1}}^{F_j}(A, \mathbf{a}_{i+1} - (2(j+1), 0), B) \leq \mu_{\Lambda_{i+1}}(A, \mathbf{a}_{i+1} - (2(j+1), 0), B).$$

This is proved using the same argument that was used for (13), except that we use a different part of the inductive hypothesis, namely, instead of using $\mu_{b(\tau_1, \tau_2)}(A, B) \leq \mu_{b(\tau_3, \tau_2)}(A, B)$, we now use $\mu_{b(\tau_1, \tau_3)}(A, B) \leq \mu_{b(\tau_1, \tau_2)}(A, B)$. Hence (17) will follow from

$$\mu_{\Lambda_{i+1}}(A, \mathbf{a}_{i+1} - (2(j+1), 0), B) \leq \mu_{\Lambda_i}(A, \mathbf{a}_{i+1}, B). \quad (18)$$

But this last inequality can be proved in the same way we proved (16) for $j = 1$. This completes the proof of (11) for $i = 0, \dots, n-2$.

We now show that (11) also holds for $i = n-1$. Unlike what we did for $i < n-1$ we cannot use the inductive hypothesis to prove (13) (see the step there that involves (14) where we needed $i \leq n-2$). We may assume that $\mathbf{a}_n \in B$ and that $\Gamma_{\Lambda_{n-1}}(A, \mathbf{a}_{n-1} - (2, 0), \mathbf{a}_n)$ is nonempty since otherwise (11) is trivial for $i = n-1$. We let

$$F = \{\text{there is no open path from } A \text{ to } B \text{ in } \Lambda_n\},$$

and note that $\mu_{\Lambda_{n-1}}^{F^c}(A, B) = \mu_{\Lambda_n}(A, B)$ and

$$\mu_{\Lambda_{n-1}}^F(A, B) = \mu_{\Lambda_{n-1}}^F(A, \mathbf{a}_{n-1} - (2, 0), \mathbf{a}_n).$$

Therefore (11) for $i = n - 1$ will follow from:

$$\mu_{\Lambda_{n-1}}^F(A, \mathbf{a}_{n-1} - (2, 0), \mathbf{a}_n) \geq \mu_{\Lambda_n}(A, B). \quad (19)$$

This inequality will follow from the following two inequalities:

$$\mu_{\Lambda_{n-1}}^F(A, \mathbf{a}_{n-1} - (2, 0), \mathbf{a}_n) \geq \mu_{\Lambda_{n-1}}(A, \mathbf{a}_{n-1} - (2, 0), \mathbf{a}_n), \quad (20)$$

and

$$\mu_{\Lambda_{n-1}}(A, \mathbf{a}_{n-1} - (2, 0), \mathbf{a}_n) \geq \mu_{\Lambda_n}(A, B). \quad (21)$$

To prove (20), note that

$$\mu_{\Lambda_{n-1}}(A, \mathbf{a}_{n-1} - (2, 0), \mathbf{a}_n) = \mu_{\Lambda_{n-1}}(A, \mathbf{a}_{n-1} - (2, 0)) \times \mu(\mathbf{a}_{n-1} - (2, 0), \mathbf{a}_n),$$

and since the event that there is an open path from $\mathbf{a}_{n-1} - (2, 0)$ to \mathbf{a}_n is independent of F , we also have

$$\mu_{\Lambda_{n-1}}^F(A, \mathbf{a}_{n-1} - (2, 0), \mathbf{a}_n) = \mu_{\Lambda_{n-1}}^F(A, \mathbf{a}_{n-1} - (2, 0)) \times \mu(\mathbf{a}_{n-1} - (2, 0), \mathbf{a}_n).$$

So by Lemma 2.2, (20) is equivalent to the inequality

$$\mu_{\Lambda_{n-1}}^F(A, \mathbf{a}_{n-1} - (2, 0)) \geq \mu_{\Lambda_{n-1}}(A, \mathbf{a}_{n-1} - (2, 0)),$$

which follows from the inductive hypothesis and the argument that was used to prove the similar inequality (14).

It remains to prove (21). This proof is very similar to the proof of (15). Namely, we define the sets

$$C_j = \{\mathbf{a}_{n-1} - (2j, 0), \dots, \mathbf{a}_{n-1} - (2, 0)\},$$

where j ranges from \underline{j} to \bar{j} , where \underline{j} is the lowest value of j for which $\Gamma_{\Lambda_n}(A, C_{\underline{j}}, B)$ is nonempty and \bar{j} is the largest value of j for which C_j is strictly to the right of τ_1 .

Then $\mu_{\Lambda_n}(A, B) = \mu_{\Lambda_n}(A, C_{\bar{j}}, B)$, so it is enough to prove by induction on j that

$$\mu_{\Lambda_{n-1}}(A, \mathbf{a}_{n-1} - (2, 0), \mathbf{a}_n) \geq \mu_{\Lambda_n}(A, C_j, B) \quad (22)$$

for each $j = \underline{j}, \dots, \bar{j}$. The case $j = \underline{j}$ is handled just as before, using the inductive hypothesis, Lemma 2.2 and Corollaries 3.1 and 3.2. The proof for the remaining values of j is similar to but simpler than that part of the argument in the proof of (15). It is simpler because the second marginal of the measure on the left of (22) is deterministic. It is slightly different because the point $\mathbf{a}_{n-1} - (2, 0)$ is an element of the set Λ_n , but this difference does not cause any difficulties. □

Now that Proposition 3.1 is proved, we can use the corollaries to Lemma 3.1 to improve the proposition by removing the restriction on the set A :

Corollary 3.4 *Modify the hypotheses of Proposition 3.1 as follows: remove the assumption that the set A strictly lie to the right of τ_3 in the first part, and the assumption that A lie strictly to the left of τ_3 in the second part. Then under this modified hypothesis, the conclusions of Proposition 3.1 remain valid.*

Proof of Corollary 3.4. We will show that $\mu_{b(\tau_1, \tau_3)}(A, B) \leq \mu_{b(\tau_1, \tau_2)}(A, B)$ without the assumption that A is strictly to the left of τ_3 . The proof of the other case is similar. Let

$$A_1 = \{\mathbf{a} \in A : \mathbf{a} \text{ is strictly to the left of } \tau_3\}$$

and

$$A_2 = A \setminus A_1.$$

Then,

$$\begin{aligned} \mu_{b(\tau_1, \tau_3)}(A, B) &= \mu_{b(\tau_1, \tau_3)}(A_1, B) \leq \\ \mu_{b(\tau_1, \tau_2)}(A_1, B) &\leq \mu_{b(\tau_1, \tau_2)}(A_1 \cup A_2, B) = \\ \mu_{b(\tau_1, \tau_2)}(A, B), \end{aligned}$$

where the first inequality follows from Proposition 3.1 and the second inequality follows from Corollary 3.1. \square

We now proceed to prove Theorem 1.1. We only prove the first of the four inequalities stated since the proof of each of the other three is similar.

Proof of Theorem 1.1. Suppose G is such that there exists at least one path from A to B in $\ell(G)$. Then $P_1(G)$ is bounded below. Now, let τ_G be the path provided by Lemma 2.1. Then,

$$\mu_{\ell(G)}(A, B) = \mu_{b(-\infty, \tau_G)}(A, B) \leq \mu_{b(-\infty, \infty)}(A, B) = \mu(A, B),$$

where the inequality follows from Corollary 3.4. \square

4 Alternative proof of Theorem 1.1

In this section we give another proof of Theorem 1.1, based on a Markov Chain introduced in [2]. Assume $A \subset L_0$, $B \subset L_n$ and E is a set of oriented edges in Λ containing at least one path from A to B . Now, let $S = \{0, 1\}^E$. Each element η of S determines the state of the edges in E in the natural way: $e \in E$ is open (closed) for η if $\eta(e) = 1$ ($\eta(e) = 0$). Now, we let T be

the subset of S consisting of the elements for which there is an open path from A to B . For $\eta \in T$, we let $\gamma_\ell(\eta)$ ($\gamma_r(\eta)$) be the leftmost (rightmost) open path from A to B under configuration η . We let $\sigma(S)$ be the σ -algebra of all subsets of S and we let P_p be the product probability measure on $\sigma(S)$ whose marginals are Bernoulli with parameter p . On the probability space $(S, \sigma(S), P_p)$, we define for each $e \in E$ a random variable X_e by means of $X_e(\eta) = \eta(e)$.

On the event T we define Γ_ℓ (Γ_r) as the leftmost (rightmost) open path from A to B . For a path γ from A to B we let $\sigma_r(\gamma)$ be the σ -algebra generated by $\{X_e : e \in \gamma \text{ or } e \text{ is to the right of } \gamma\}$ and we let $\sigma'_r(\gamma)$ be the σ -algebra generated by $\{X_e : e \notin \gamma, e \text{ is to the right of } \gamma\}$. Similarly we let $\sigma_\ell(\gamma)$ be the σ -algebra generated by $\{X_e : e \in \gamma \text{ or } e \text{ is to the left of } \gamma\}$ and we let $\sigma'_\ell(\gamma)$ be the σ -algebra generated by $\{X_e : e \notin \gamma, e \text{ is to the left of } \gamma\}$. We now note that the event $\{\Gamma_\ell = \gamma\}$ is $\sigma_\ell(\gamma)$ -measurable. Therefore under the conditional measure $P_p(\bullet | \Gamma_\ell = \gamma)$ the distribution of the state of the bonds which are strictly to the right of γ remains a Bernoulli product measure of parameter p . Similarly, under the conditional measure $P_p(\bullet | \Gamma_r = \gamma)$ the distribution of the state of the bonds which are strictly to the left of γ remains a Bernoulli product measure of parameter p . We now define a Markov Chain on T . Its transition mechanism is given in two steps. For a given initial state η_0 , first we choose $\eta_{1/2} \in T$ by letting $\eta_{1/2}(e) = \eta_0(e)$ for all e to the left of $\gamma_\ell(\eta_0)$ or on $\gamma_\ell(\eta_0)$ and for the other elements of E we let $\eta_{1/2}(e)$ be independent Bernoulli random variables with parameter p . Once we have determined $\eta_{1/2}$ we let $\eta_1(e) = \eta_{1/2}(e)$ for all e to the right of $\gamma_r(\eta_{1/2})$ or on $\gamma_r(\eta_{1/2})$ and for the other elements of E we let $\eta_1(e)$ be independent Bernoulli random variables with parameter p which are also independent of the random variables used to determine $\eta_{1/2}$. In the sequel we will need to consider this Markov Chain for different sets E . We will call it the Markov Chain associated to E . In the sequel, we extend the notation of the previous sections by letting $\Gamma_E(A, B)$ be the set of paths from A to B whose edges are in E .

Proposition 4.1 (*van den Berg, Häggström, Kahn*) *The measure $P_p(\bullet | T)$ is invariant for the Markov chain.*

Proof of Proposition 4.1. We show that if the initial state of the chain η_0 is chosen according to the distribution $P_p(\bullet | T)$, then $\eta_{1/2}$ has the same distribution. A similar argument will then show that η_1 has the same distribution as $\eta_{1/2}$. Let γ be an arbitrary path in $\Gamma_E(A, B)$ and let

$$S_\gamma = \{\eta \in T : \gamma \text{ is the leftmost open path from } A \text{ to } B\}$$

under configuration s }.

Then $(S_\gamma : \gamma \in \Gamma_E(A, B))$ is a partition of T and $P_p(\bullet|T)$ is a convex combination of the measures $(P_p(\bullet|S_\gamma))_{\gamma \in \Gamma_E(A, B)}$. Therefore, it suffices to show that if η_0 is distributed according to some $P_p(\bullet|S_\gamma)$ then $\eta_{1/2}$ is also distributed according to that measure. But this is an immediate consequence of the way we obtain $\eta_{1/2}$ from η_0 and the already observed fact that the under the conditional measure $P_p(\bullet|\Gamma_\ell = \gamma)$ the distribution of the state of the bonds which are strictly to the right of γ remains a Bernoulli product measure of parameter p .

□

Since the Markov Chain associated to E is obviously irreducible and aperiodic we deduce from this proposition the following:

Corollary 4.1 *From any initial distribution, the Markov Chain associated to E converges to $P_p(\bullet|T)$*

Proof of Theorem 1.1. As before, we only prove the first inequality, since the other proofs are similar. Let E be the set of edges belonging to paths in $\Gamma(A, B)$ and let $E_{\ell(G)}$ be the set of edges belonging to paths in $\Gamma_{\ell(G)}(A, B)$. We let T_1 be the subset of elements of $\{0, 1\}^E$ for which there exists an open path from A to B and we T_2 be the subset of elements of $\{0, 1\}^{E_{\ell(G)}}$ for which there exists an open path from A to B . We now construct a Markov Chain in

$$X = \{(\eta, \xi) \in T_1 \times T_2 : \gamma_\ell(\eta) \geq \gamma_\ell(\xi), \gamma_r(\eta) \geq \gamma_r(\xi)\}$$

whose first and second marginals are as the Markov Chains associated to E and to $E_{\ell(G)}$ respectively. This is done as follows: Assume $(\eta_0, \xi_0) \in X$, then let $\{Y_e : e \text{ strictly to the right of } \gamma_\ell(\xi)\}$ be a collection of i.i.d Bernoulli random variables of parameter p . First note by definition of X , $\gamma_\ell(\eta) \geq \gamma_\ell(\xi)$, then define $\eta_{1/2}$ and $\xi_{1/2}$ as follows:

$$\begin{aligned} \eta_{1/2}(e) &= \eta_0(e) \text{ for all } e \text{ on } \gamma_\ell(\eta_0) \text{ or to the left of } \gamma_\ell(\eta_0), \\ \eta_{1/2}(e) &= Y_e \text{ for all } e \text{ strictly to the right of } \gamma_\ell(\eta_0), \\ \xi_{1/2}(e) &= \xi_0(e) \text{ for all } e \text{ on } \gamma_\ell(\xi_0) \text{ or to the left of } \gamma_\ell(\xi_0), \\ \xi_{1/2}(e) &= Y_e \text{ for all } e \text{ strictly to the right of } \gamma_\ell(\xi_0). \end{aligned}$$

After that note that $(\eta_{1/2}, \xi_{1/2}) \in X$ and let $\{Z_e : e \text{ strictly to the left of } \gamma_r(\eta)\}$ be collection of i.i.d Bernoulli random variables of parameter p which is independent of the random variables X_e . Finally, define η_1 and ξ_1 as follows:

$$\begin{aligned} \eta_1(e) &= \eta_{1/2}(e) \text{ for all } e \text{ on } \gamma_r(\eta_{1/2}) \text{ or to the right of } \gamma_r(\eta_{1/2}), \\ \eta_1(e) &= Z_e \text{ for all } e \text{ strictly to the left of } \gamma_r(\eta_{1/2}), \\ \xi_1(e) &= \xi_{1/2}(e) \text{ for all } e \text{ on } \gamma_r(\xi_{1/2}) \text{ or to the right of } \gamma_r(\xi_{1/2}), \end{aligned}$$

$\xi_1(e) = Z_e$ for all e strictly to the left of $\gamma_r(\xi_{1/2})$.

We can now complete our proof: let Φ be a bounded increasing function on $\Gamma(A, B)$ and let (η_0, ξ_0) be an element of X . Then $(\eta_n, \xi_n) \in X$ a.s. $\forall n$. Therefore,

$$\Phi(\gamma_\ell(\eta_n)) \geq \Phi(\gamma_\ell(\xi_n)) \text{ a.s. } \forall n.$$

Hence,

$$E(\Phi(\gamma_\ell(\eta_n))) \geq E(\Phi(\gamma_\ell(\xi_n))) \quad \forall n,$$

and applying Corollary 4.1 to both sides of the inequality we get $\mu(A, B) \geq \mu_{\ell(G)}(A, B)$. □

5 Generalizations and extensions of Theorem 1.1

We first discuss generalizations of the oriented bond percolation model that do not require any change in our proofs of the main results, except for one case in which the Markov chain proof does not seem to work. Then we consider oriented site percolation and the contact process.

It is easy to check that we never made any use of the assumption that all of the bonds have the same probability of being open. In fact, we could assign a different probability to each bond, and the proofs will continue to work without any changes. It may seem that this is an uninteresting generalization, but we will see that it turns out to be relevant when we use percolation models to approximate the continuous time contact process.

Another easy generalization involves the assumption of independence between bonds. Nowhere in our original proofs of the main result do we need the openness of two bonds to be independent if those two bonds emerge from the same site. That is, if (x, y) is a site in Λ , then the events that the two bonds that connect (x, y) to $(x \pm 1, y + 1)$ are open can be correlated arbitrarily. Independence is only needed for bonds that emerge from different sites. See the comment that is found in the proof of Proposition 3.1, at the end of the paragraph where the random set Φ is defined. This is one situation where the Markov chain proof does not work quite so well; it does not seem to be valid in the case of negative correlations.

We now turn to oriented site percolation. Let

$$\overline{\Lambda} = \{(x, y) : x, y \in \mathbb{Z}, y \geq 0\}.$$

Fix integers $a \leq 0 < b$, and for each $(x, y) \in \overline{\Lambda}$, introduce oriented bonds from (x, y) to $(x + k, y + 1)$ for $a \leq k \leq b$. All of the bonds are open. The

sites are open independently of each other, with probability $p \in (0, 1)$ (we could also allow different probabilities for different sites). Paths are defined in the obvious way, and open paths are paths in which all of the sites are open.

The case where $a = 0$ and $b = 1$ is equivalent to the standard oriented site percolation in \mathbb{Z}^2 , where there are two bonds per site. Not surprisingly, our proofs of the main result can be modified in a routine way to cover this case. The only real difference involves the places in the proofs of Lemma 3.1 and Proposition 3.1 where a random set Φ is defined. In the case of oriented bond percolation, Φ was defined in each proof to be a certain random set of open edges belonging to paths that either started in some set A or ended in a set B . For the oriented site percolation model, it is best to define Φ to be an analogous random set of open sites, and then enlarging Φ to include all of the (typically closed) sites that are at the ends of bonds that are connected to sites in Φ . Then the proofs can be continued as before, looking at open paths that are strictly to the right of the enlarged version of Φ .

Once the case $a = 0$, $b = 1$ is handled, it is quite routine to further modify the proof to cover arbitrary $a \leq 0 < b$, which is to say that our main result holds for finite range oriented site percolation in 2 dimensions. This highlights a significant difference between oriented bond percolation and oriented site percolation. In oriented bond percolation, the leftmost and rightmost paths may not even exist when there is the possibility that bonds cross one another, as will be the case when $b - a > 1$. But in oriented site percolation, leftmost and rightmost paths always exist, for any choice of a, b .

Finally, we briefly discuss the contact process. By treating the y -coordinate in Λ as the time variable, one can obtain various versions of the discrete time contact process from oriented percolation. The standard model is equivalent to oriented site percolation. Variations on this model can be obtained from oriented bond percolation, and also by looking at mixed models in which both the sites and the bonds can be open or closed. Not surprisingly, our main results applies to many mixed percolation models, and hence to many different discrete time contact processes.

One way to extend our results to the continuous time contact process is to approximate continuous time with discrete time. This method works easiest for the one-sided nearest neighbor contact process. Then we could use the oriented bond percolation model that is the setting for most of this paper, but it is perhaps more natural to do oriented bond percolation on an equivalent graph: the set of sites is $\bar{\Lambda}$ and the oriented bonds are those that correspond to $a = 0$ and $b = 1$. That is, two oriented bonds emerge from each site (x, y) , a “vertical” bond connecting it to the site $(x, y + 1)$ and a “contact” bond connecting it to $(x + 1, y + 1)$.

With this setup, we can approximate the continuous time one-sided contact process by letting the contact bonds be open with small probability $h > 0$ and letting the vertical bonds be open with probability $1 - \varepsilon h$, where $\varepsilon \geq 0$ is a parameter of the model. Then letting $h \downarrow 0$ and rescaling time by a factor of h produces the continuous-time model. This shows why it can be desirable to allow the bonds to have different probabilities of being open.

One can use a similar approximation method for the two-sided nearest neighbor contact process. In this case, we use the graph Λ that is the setting for the bulk of this paper, but we add additional oriented “vertical bonds” that connect each site $(x, y) \in \Lambda$ to the site $(x, y + 2)$. Then by assigning appropriate probabilities to the bonds (different for the vertical bonds than for the diagonal bonds), one obtains a percolation model that depends on a parameter h , and this model converges to the two-sided nearest neighbor contact process as $h \rightarrow 0$. See [1] for further details. The bottom line is that the results in this paper all apply to one the one-sided and two-sided contact processes in continuous time.

6 Applications of Theorem 1.1

One reason for our interest in extreme paths is that they provide us with a useful way to analyze various conditional probabilities, and with the help of Theorem 1.1, we are able to make comparisons that go beyond the usual correlation inequalities that are familiar in percolation theory. The results in this section apply to the more general models discussed in the previous section, except that Corollary 6.2 requires translation invariance, so that all of the bond (or site) probabilities must be the same.

We will rely on a key fact about extreme paths. It is that if γ is a path, the event that γ is a rightmost (leftmost) extremal path is measurable with respect to the states of the bonds that are to the right (left) of γ and hence this event is independent of the states of the bonds that are strictly to the left (right) of γ . (For more details, see the discussion and definitions at the beginning of the previous section.)

The following result and its proof show how we use this fact in conjunction with Theorem 1.1 to compare several different conditional probabilities.

Lemma 6.1 *Let $n > 0$, let A be a finite subset of L_0 and let B_1, B_2, B_3 be finite subsets of L_n . Suppose that B_1 is strictly to the left of B_2 and that B_2 is strictly to the left of B_3 . For $i = 1, 2, 3$, define the events*

$$H_i = \{\text{there exists an open path from } A \text{ to } B_i\}$$

If the events H_2 and H_3 have nonzero probability, then

$$P(H_1 \mid H_2 \cap H_3^c) \geq P(H_1 \mid H_2) \geq P(H_1 \mid H_2 \cap H_3) \quad (23)$$

and

$$P(H_1 \mid H_2) \geq P(H_1 \mid H_3). \quad (24)$$

Furthermore, if A consists of a single site $(x, 0)$, then $P(H_1 \mid H_2)$ is nonincreasing in x for all x such that the event H_2 has positive probability.

Proof of Lemma 6.1. We begin by proving the following inequality:

$$\nu^{H_3^c}(A, B_2) \leq \nu(A, B_2). \quad (25)$$

The proof of (25) is similar to arguments that we have made before. The event H_3^c is the disjoint union of events of the form $\{\Phi = \varphi\}$, where Φ is the random set of all edges that are contained in open paths that end in B_3 . The left side of (25) is a convex combination of the measures $\nu^{\{\Phi = \varphi\}}(A, B_2)$, and for each φ , $\nu^{\{\Phi = \varphi\}}(A, B_2) = \nu_{\ell(G)}(A, B_2)$, where G is the set that contains the endpoints of the edges in φ . The inequality in (25) now follows from Theorem 1.1.

We now use (25) to prove the first inequality in (23). Let Γ be the rightmost open path from A to B_2 , assuming that such a path exists, which is the same as assuming that H_2 occurs. The left side of (25) is the conditional distribution of Γ given $H_2 \cap H_3^c$ and the right side of (25) is the conditional distribution of Γ given H_2 . In either case, the event H_1 occurs if and only if there is an open path from $A \cup \Gamma$ to B_1 .

Let γ be any path from A to B_1 . Given the event $H_2 \cap \{\Gamma = \gamma\}$, the edges strictly to the left of γ are each open with probability p and they are independent of each other. This statement about the edges left of γ also holds true given the event $H_2 \cap H_3^c \cap \{\Gamma = \gamma\}$ because of the assumption that H_3 is strictly to the right of H_2 . In either case, whether or not there is an open path from $A \cup \gamma$ to B_1 is determined in the same way by the openness of the edges that are strictly to the left of γ . Thus, there is a function φ on the set of all paths γ from A to B_2 such that

$$\varphi(\gamma) = P(H_1 \mid H_2 \cap \{\Gamma = \gamma\}) = P(H_1 \mid H_2 \cap H_3^c \cap \{\Gamma = \gamma\}),$$

and we have

$$E(\varphi(\Gamma) \mid H_2) = P(H_1 \mid H_2) \quad \text{and} \quad E(\varphi(\Gamma) \mid H_2 \cap H_3^c) = P(H_1 \mid H_2 \cap H_3^c).$$

Clearly, φ is monotone with respect to the partial ordering on paths, so The first inequality in (23) now follows from (25). The second inequality in (23) follows immediately from the first inequality and the fact that the middle expression in (25) is a convex combination of the first and third expressions.

To prove (24), we note that repeated applications of Corollary 1.1 imply that

$$\nu(A, B_2) \leq \nu(A, B_3).$$

(See the proof of Corollary 3.1 for a similar argument involving repeated applications of Corollary 1.1.) Now (24) follows, in the same way that (23) followed from (25).

To prove the last part of the lemma, we note that by two applications of Corollary 1.1,

$$\nu((x, 0), B_2) \leq \nu((x + 2, 0), B_2)$$

for all x such that there exist paths from $(x, 0)$ and $(x + 2, m)$ to B_2 . The last part of the lemma is now proved in the same way that (23) and (24) were proved. \square

The first inequality in (23) may seem counterintuitive. We know that the occurrence of a “negative” event like H_e^c makes a “positive” event like H_1 less likely to occur. But the first inequality says informally that once H_2 occurs, the additional occurrence of the negative event H_3^c makes H_1 *more likely* to occur. Here is another way to state this surprising result:

Corollary 6.1 *Let H_1, H_2, H_3 be as in Lemma 6.1. Then given H_2 , the events H_1 and H_3 are conditionally negatively correlated.*

Proof of Corollary 6.1. The proof is elementary, using the second inequality in Lemma 6.1:

$$\begin{aligned} P(H_1 \cap H_3 \mid H_2) &= \frac{P(H_1 \cap H_2 \cap H_3)}{P(H_2)} = \frac{P(H_1 \cap H_2 \cap H_3)}{P(H_2 \cap H_3)} P(H_3 \mid H_2) \\ &= P(H_1 \mid H_2 \cap H_3) P(H_3 \mid H_2) \leq P(H_1 \mid H_2) P(H_3 \mid H_2). \end{aligned}$$

\square

Here is another application of Lemma 6.1. It is a rather natural monotonicity involving certain percolation probabilities. It is somewhat surprising that its proof seems to require the consequences of something as sophisticated as Theorem 1.1. We note that this result clearly depends on some translation invariance, so it requires all of the bond probabilities (or site probabilities

in the case of oriented site percolation) to be the same. A different proof of this result is given in [1], but we believe the one given here is more natural and easier to follow.

Corollary 6.2 *Let $0 \leq m < n$ and let x, y be integers such that $(x, m) \in L_m$ and $(y, n) \in L_n$. Let $A_{x,y}$ be the event that there is an open path from (x, m) to (y, n) . Then $P(A_{x,y})$ is nonincreasing in $|x - y|$.*

Proof of Corollary 6.2. Because of the natural symmetries built into the percolation model, we may assume without loss of generality that $m = 0$, $x = 0$ and $y \geq 0$. The obvious inductive argument reduces the proof to showing that

$$P(A_{0,y}) \geq P(A_{0,y+2}), \quad (26)$$

where we may assume that y is such that there exists at least one path from $(0, 0)$ to $(n, y+2)$. Since $y \geq 0$, this assumption implies that there also exists at least one path from $(0, 0)$ to (n, y) .

Under these circumstances, to prove (26), it is enough to prove that

$$P(A_{0,y} \mid A_{0,y+2}) \geq P(A_{0,y+2} \mid A_{0,y}) \quad (27)$$

since the numerators in the expressions for the two conditional probabilities in (27) are the same and since the denominators in these expressions are the two sides of (26) (in reverse order).

To prove (27), we first use left-right symmetry and then translation invariance to get

$$P(A_{0,y+2} \mid A_{0,y}) = P(A_{0,-y-2} \mid A_{0,-y}) = P(A_{2y+2,y} \mid A_{2y+2,y+2}).$$

The last part of Lemma 6.1 implies that we make the right side of this equation no smaller if we replace $2y+2$ by $2y$, then by $2y-2$, then by $2y-4$, and so on. Since we assumed that $y \geq 0$, this gives us

$$P(A_{2y+2,y} \mid A_{2y+2,y+2}) \leq P(A_{0,y} \mid A_{0,y+2})$$

proving (27), and thus (26). Note that our assumptions about y ensure that all of the relevant events in these applications of Lemma 6.1 have positive probability, as required by the hypotheses of that lemma. \square

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